Relative Probable Error

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Can I Use These Measurements?

The answer to the question above depends upon what you want to do with the measurements.

- If you want to use the measurements *to monitor a production process for possible changes*, the answer is given in the article "Good Data, Bad Data, and Process Behavior Charts."
- If you want to use the measurements *to characterize product as either conforming or nonconforming*, the answer is given in the article "How to Establish Manufacturing Specifications."
- But if you want to use measurements *to quantify some characteristic of an item*, then the answer to the question above will depend upon the *Relative Probable Error*.

Probable Error

The use of Probable Error to quantify the precision of a predictable measurement system was carefully developed in Sections 6 and 7 of "How to Establish Manufacturing Specifications." There it was shown that the Probable Error defines the effective resolution of a measurement and determines the appropriate size of the measurement unit. The simplest way to estimate the Probable Error for any measurement system is to obtain repeated measurements of a single item. When placed on an *XmR* Chart these repeated measurements will allow you to judge the predictability of the measurement system and to also estimate the standard deviation of the measurement system, *SD(E).* If the measurement system is reasonably predictable, one estimate of *SD(E)* would be:

$$\hat{\sigma}_e = \frac{\bar{R}}{d_2}$$

Once you have an estimate of *SD(E)*, the Probable Error is defined to be:

$$P.E. = 0.675\ \hat{\sigma}_e$$

The Probable Error defines just how large your measurement increments should be. Ideally your measurement increment would be exactly the same size as the Probable Error. In practice, since we only have an estimate of the Probable Error, and since writing one extra digit will change the measurement increment by an order of magnitude, we have to allow some lee-way. Both experience and theory agree, however, that the measurement increment needs to be approximately the same size as the Probable Error. Thus, in practice, you should seek to have:

$$0.20\ \textit{Probable Error} \le \text{Measurement Increment} \le 2\ \textit{Probable Error}$$

Recording your measurements to a precision greater than 0.2 *P.E.* is excessive. The extra digits recorded will not contain any real information about the item being measured. On the other hand, rounding your measurements off to units that are larger than 2 *P.E.* is wasteful—you are throwing away useful information about the item that was measured.

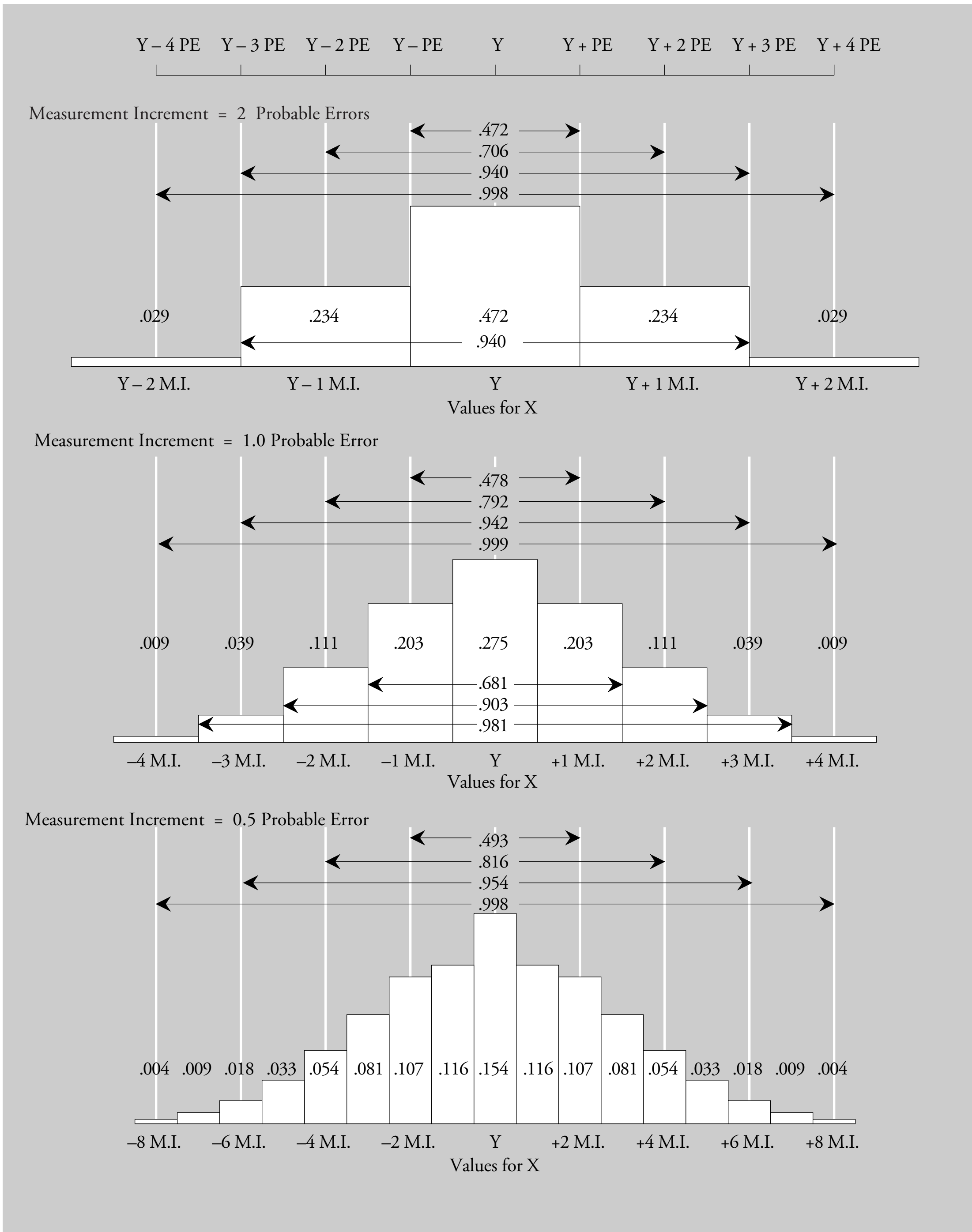

Figure 1: The Probabilities of Different Values for *X* for a Given Value of *Y*

If we let *X* denote the product measurements, let *Y* denote the product values, and let *E* denote the measurement error, then $X = Y + E$. Figure 1 shows the relationship between *X* and *Y* for three different choices for the measurement increment. If the measurement increment is equal to 2 *P.E.* and *Y* is rounded off to the nearest measurement increment, then *X* will be equal to *Y* about half the time, and *X* will differ from *Y* by one measurement increment or less about 94 percent of the time.

If the measurement increment is equal to the Probable Error, when *Y* is rounded off to the nearest measurement increment, *X* will be equal to *Y* about 27% of the time; *X* will differ from *Y* by one measurement increment or less about 68% of the time; *X* will differ from *Y* by two measurement increments or less about 90% of the time; and *X* will differ from *Y* by three measurement increments or less about 98% of the time. While these numbers do not sound as good as those in the preceding paragraph, Figure 1 shows that the two sets of numbers are essentially describing the same amount of uncertainty.

While the probabilities for each of the increments differs in the three graphs, the similarity of the three graphs is revealed by the probabilities shown above each graph. These probabilities of being within one, two, three, or four probable errors of *Y* are almost the same for all three graphs. Using a smaller measurement increment does not reduce the uncertainty in you r measurement, it just spreads the same amount of uncertainty over more values. Using a larger measurement increment may increase the certainty in the last recorded digit, but it does so at the expense of increased chunkiness and degraded measurements. (This effect of larger measurement increments can be seen in two ways in Figure 1. Not only is the top graph slightly wider than the others, but also we see that the probability that *X* will be within one *P.E.* of *Y* is smaller than with the other graphs. These effects, and others like them, will become more pronounced as larger increments are used.)

Thus, the concept of Probable Error allows us to go directly from an estimate of the uncertainty of the measurement process to a clear statement of what sized measurement increments are appropriate for a given measurement system. We will make use of this knowledge later.

The Coefficient of Variation

A traditional measure of the relative amount of variation in a particular measurement is the *Coefficient of Variation for a Measurement:*

$$\text{Coefficient of Variation for a Measurement} = CVM = \frac{Est.\ SD(E)}{\text{Average Value for Typical Measurements}}$$

where *Est. SD(E)* is the estimated standard deviation of the measurement system. The Coefficient of Variation for a Measurement is used in many fields to characterize the quality of a measurement and various guidelines for what is acceptable exist. Obviously smaller values indicate better measurements, and larger values are associated with weaker measurements.

Say, for example, that you are making paving blocks that are supposed to be six inches by nine inches. If your measurement system has an estimated standard deviation of 0.030 inches, then the Coefficient of Variation for the measurements of the long side would be 0.0033 or 0.33%, while that for the short side would be 0.0050 or 0.50%. While these Coefficients of Variation are easy to compute, and while they allow comparisons and evaluations of the quality of various measurements, it is hard to put in words exactly what such numbers represent. On the other hand, the fact that the Probable Error is 0.020 inches tells us that we should not record these values any more precisely than to the nearest 1/50 of an inch.

The Relative Probable Error

The problem of expressing just what is measured by the Coefficient of Variation for a Measurement (*CVM*) comes from the problem of describing a standard deviation in words. However, since the Probable Error is function of the standard deviation, and since the Probable Error has a simple explanation, we can remedy the inarticulate nature of the Coefficient of Variation by transforming it into the *Relative Probable Error*. The Relative Probable Error is the Probable Error of a measurement divided by a typical measurement value:

$$\textit{Relative Probable Error} = \frac{0.675 \; \textit{Est. SD(E)}}{\text{Average Value of Measurements}} = 0.675 \; \textit{CVM}$$

The Relative Probable Error compares the effective resolution of the measurements with the size of the measurements, thus it is the inverse of the effective number of measurement increments contained in a typical measurement. When the *RPE* is 0.01 (one percent) the measurements have the potential of being known to one part per hundred when the measurement increment is sized correctly.

Figure 2 shows the number of effective measurement units contained in a typical measurement for eight different values of the Relative Probable Error. Also shown are the corresponding Coefficients of Variation for the Measurement.

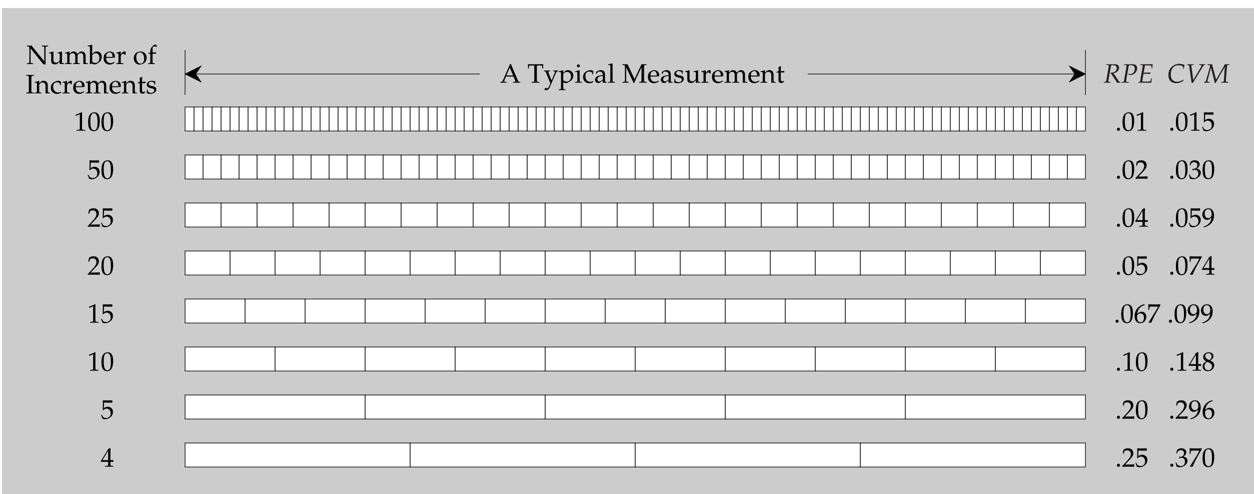

Figure 2: The Effective Precision for a Measurement is Described by the Relative Probable Error

When the *RPE* is 0.02 the measurements have the potential to be known to the nearest two percent. Such measurements will err by less than two percent about half the time, and they will err by more than two percent about half the time. While a measurement increment that is smaller than the Probable Error may make the measurements look like they are known to better than the nearest two percent, interpreting them more precisely than to the nearest two percent would be a mistake. On the other hand, if the measurement increment exceeds the Probable Error, a measurement with an *RPE* of 0.02 may have fewer than 50 increments in a typical value. When this happens the round-off will determine the effective resolution and the measurement may not be known to the nearest two percent. (To express the effective resolution when the measurement increment dominates the probable error, you could divide the measurement increment by a typical average value and use this number in place of the Relative Probable Error.)

For the paving blocks in the previous section, the measurement system has an *RPE* of 0.0034 for the short side of the blocks. If the measurement increment is less than or equal to the Probable Error, which is 0.675 (0.030″) = 0.02 inch, the measurements of the short side have the potential of being known to a precision of (1/*RPE* = 1/0.0034) or one part in 300.

If the measurements are made with the measurement system characterized above, but are only recorded to a sixteenth of an inch (0.0625″) the round-off will be three times the Probable Error, and the round-off will define the precision of the measurements In this case the measurements of the short side will still have a *Relative Precision* of 0.0625″/6.0″ = 0.01 or one part in a hundred. If this degree of precision is sufficient, then you do not need to be concerned about the measurement units being too large. If greater precision is desired, this measurement system has the ability to deliver better precision—but you will have to record the values using smaller measurement increments in order to realize the full potential of this measurement system.

For another example consider the following inspection problem. A special edition of a textbook was produced at a quick-print shop. Some of the 250 copies produced were found to have missing pages while others had extra pages. Thus the problem was how to inspect these 404-page books to find the ones with extra pages or missing pages. A 404-page book has 202 sheets of paper. In this case, the soft-bound books had an average overall thickness of 0.954 inches or 954 mils. Therefore, to detect a missing page, they needed to be able to detect a difference in the thickness of 954/202 = 4.7 mils. Using a standard vernier caliper, the thickness of a book could be measured. Twenty measurements of the thickness taken at the top of a single book were collected and placed on the process behavior chart in Figure 3.

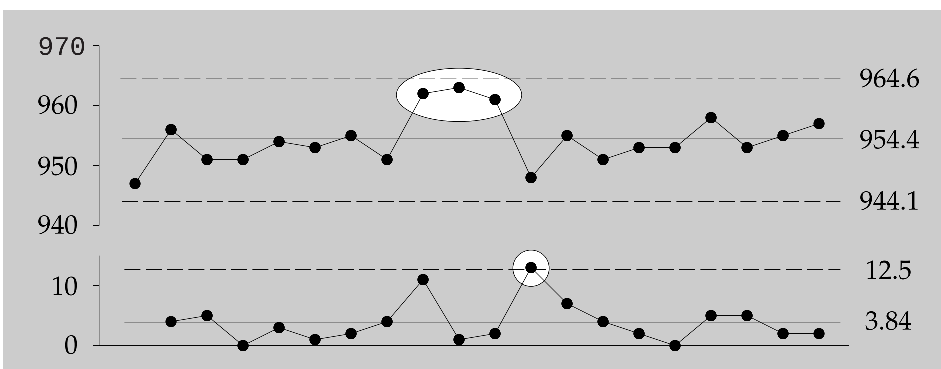

Figure 3: *XmR* Chart for Repeated Measurements of Book Thickness

The signals of exceptional variation seen on the *XmR* Chart in Figure 2 were unexpected. They suggest a problem with the consistency of the measurement procedure. Upon investigation it was discovered that these books were thicker near the spine (possibly due to the glue). Originally no special care had been exercised to measure the book at the same spot. A second set of 20 measurements of the thickness of one book were collected with care taken to measure near the middle of the top edge of the book. These values are shown in Figure 4.

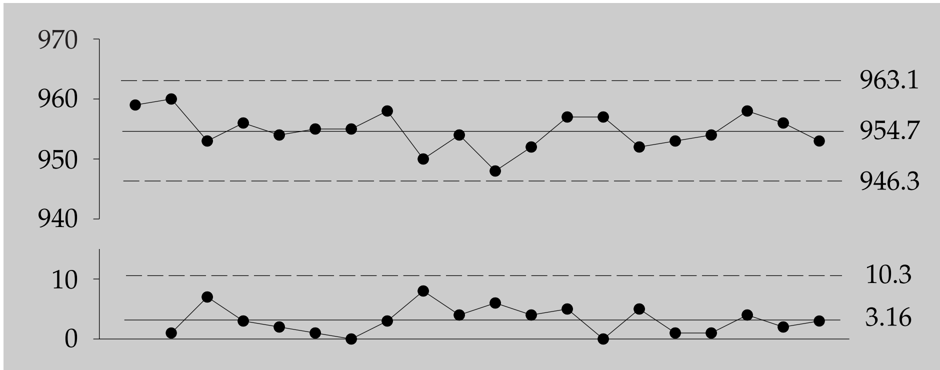

Figure 3: *XmR* Chart for Second Set of Repeated Measurements of Book Thickness

The absence of signals in Figure 4 is taken as evidence of a consistent measurement procedure. One estimate of the standard deviation of this measurement process is obtained by dividing the Average Moving Range by d_2 = 1.128. This gives an *Est. SD(E)* of 2.80, and this estimate has 12 degrees of freedom. Since the data in Figure 3 are homogeneous, we could, as an alternative, use the standard deviation statistic of s = 3.03 to obtain an *Est. SD(E)* having 19 degrees of freedom.

This last value for *Est. SD(E)* gives a Probable Error of 0.675 (3.03) = 2.04 mils. Thus, using the vernier caliper to measure the thickness of the books results in an *RPE* of 2.04/954.7 = 0.002. The Probable Error tells us that these measurements will essentially be known to the nearest 2 mils, while the *RPE* indicates that these measurements will be known to the precision of 2 parts per thousand, or one part in 500.

Measuring the thickness of these books to within one part in 500 is sufficient to detect missing sheets or extra sheets in books with 202 sheets. Thus, once the inconsistencies in the use of the vernier caliper were removed it provided measurements that were adequate for this job.

In the preceding example the situation defined the precision needed in the measurements. In other situations a specific level of precision may not be required. The Relative Probable Error will still tell you just how good a given set of measurements happens to be.

In order to carry out a special study Terry needed to periodically select 100 uncured rubber blocks from the production line and measure their lengths. Since he recognized the difficulties involved in measuring uncured rubber, he spent three days searching for a special vernier scale having a large surface area on the calipers. He used this special instrument to measure one block repeatedly and placed the repeated measurements on an *XmR* Chart. This chart showed that his measurements were consistent, but that they had a lot of variation due to the way the calipers would squeeze the rubber. Terry's estimate of *SD(E)* was 1.46 mm. Since the blocks averaged about 160 mm in length, he had a Relative Probable Error of:

$$RPE = \frac{0.675\ (1.46\text{ mm})}{160\text{ mm}} = \frac{0.98\text{ mm}}{160\text{ mm}} = 0.0062$$

A value of six parts per thousand corresponds to about one part per 167. In spite of his high-precision tool, which gave measurements to the nearest 0.01 mm, the inherent difficulties of measuring the soft rubber resulted in measurements that were only good to the nearest millimeter. At this point Terry realized that he probably did not need the special vernier calipers—he could get measurements that were good to the nearest millimeter using a steel scale and the naked eye.

Notice that in the two preceding examples the *SD(E)* was estimated by actually using the measurement system to repeatedly measure a single item. This allowed for the interaction of the measurement system and the item measured to be incorporated into the analysis. In both cases the measurement device was capable of squeezing the part being measured. This squeezing degraded the measurements and therefore needed to be reflected in the estimate of *SD(E)* used in computing the Relative Probable Error.

To illustrate how the way the measuring device is used can affect the estimate of *SD(E)* I measured a granite block using the same vernier caliper that was used in the first example. Care was taken to measure the same place on the block each time, and the 20 values did not display any evidence of inconsistency when placed on an *XmR* Chart.

The average of these 20 measurements was 3.454 inches, with a standard deviation statistic of s = 0.00089 inches. Thus, for this measurement, the calipers have a Probable Error of 0.6 mils. Using the earlier guideline, this Probable Error suggests that we need a measurement increment that is somewhere between 0.20 PE = 0.12 mils and 2 PE = 1.2 mils. Thus, recording these values to the nearest whole number of mils is both appropriate and correct. The Relative Probable Error is 0.000174, which rounds off to 2 parts per ten thousand or one part in five thousand.

When the same instrument was used to measure spongy books the Probable Error was three times greater (2.0 mils versus 0.6 mils). Using the guideline, these measurements of book thicknesses should be recorded using increments that are between 0.40 mils and 4 mils. Since the scale gives values to the nearest mil, and since one mil falls in this range, it is likely that the book thicknesses would be recorded to the nearest mil. Thus, while the measurements on the granite block and the measurements of the book thicknesses are both recorded to the same number of digits, they do not have the same precision. Without a knowledge of the Probable Errors, this difference in precision would go undetected.

Since measurement systems can interact with the parts being measured, it is important to characterize the measurement error for each different application.

Summary

The Probable Error describes the effective resolution of a measurement, and determines how many digits should be recorded for that measurement.

The Relative Probable Error defines the essential uncertainty in a measurement. When the measurement increment is smaller than the Probable Error the Relative Probable Error is the inverse of the effective number of measurement units contained in a typical measurement.

When you wish to use a measurement to quantify a particular characteristic, you will usually be able to determine if that measurement will meet your needs by using the Probable Error to determine the appropriate measurement increment and the Relative Probable Error to determine the essential uncertainty of the measurement. It really is this simple.