Analysis Using Few Data

Some of these batches are not like the others …

Donald J. Wheeler

In some industries a few test batches will be produced prior to going into production. When this happens a critical question is "Are all of the test batches alike?" With only one value per batch, how can we compare a set of three or more values to see if one of the values is different from the others? This paper will answer this question by presenting a new test for homogeneity.

The problem here is substantially different from most of the problems addressed by statistical tests. Here we will want to use the test batches to characterize future production, but before we can meaningfully interpret even the simplest statistics in this way we will need homogeneity between the test batches. Say we have five test batches of a new product, and five values for one characteristic of those batches. While we can always compute an average value for this characteristic, what does this average represent? Can we use this average as a prediction for what we hope to produce in practice? It will make sense to do so only when the five values are homogeneous. If the five values are not homogeneous, then the statistic loses meaning because it does not represent any underlying reality. (Think about computing the average weight of three apples and two grapefruit. While the average may describe the five pieces of fruit, the result cannot be generalized in any meaningful way, and the key to knowing that the average value will not generalize is knowing that the five pieces of fruit are a mixture of unlike things.)

Homogeneity is implicitly assumed by virtually all statistical techniques. Before we can generalize the results of those techniques to provide the answers needed to do business, we will need to have some basis for assuring ourselves and others that the requisite homogeneity is present. This takes such chestnuts as confidence intervals, lack-of-fit tests, and transformations of the data off the table until after we have some basis for saying that the data are homogeneous. The technique presented here will provide such a basis.

PROCESS BEHAVIOR CHARTS

The simple *XmR* chart is useful in checking a continuing sequence of values for homogeneity. This is done by using the limits obtained from a baseline period to characterize future values as either being consistent, or inconsistent, with the baseline period. As I described in my March column, "Exact Answers to the Wrong Questions," we usually want anywhere from 10 to 20 values in the baseline period. (Whenever we start with fewer values, the intention is, in the absence of signals, to update the short baseline as additional data become available.)

When we are focused on the analysis of a small, finite data set rather than working with a continuing sequence of values there are some problems with using the *XmR* chart. First, when we have 4 or fewer values in the baseline period it is impossible for any of the baseline values to fall outside the limits using the usual computations. Second, when we have fewer than 8 values in the baseline, the only baseline point that can fall outside the limits of the *X* Chart will be either the first point or the last point. This means that when we use less than 8 values on an *XmR* Chart there are some types of signals that simply cannot be detected by the chart. This inability to detect some differences that might exist within the data makes the *XmR* Chart a less than satisfactory tool for the analysis of a finite data set consisting of less than 10 values. Hence, we shall have to use some other approach to analyzing finite data sets containing fewer than 10 values for evidence of a lack of homogeneity.

THE NATURE OF THE QUESTION

Anyone desperate enough to test fewer than ten values for homogeneity needs to understand the Faustian bargain they are making. With any technique for the analysis of data there will be certain fixed relationships between (1) the amount of data available, (2) the ability to detect signals within those data, and (3) the risk of getting a false alarm when there are no signals within those data. The name we give the ability to detect signals is power. The name we give to the risk of a false alarm is the alpha-level. For a fixed alpha-level, we can get more power by using more data. However, for a fixed amount of data, we can only get more power by using a larger alpha-level. Here we are going to be working at the lower end of the spectrum in terms of the amount of data available. Therefore, in order to have any reasonable ability to detect signals within our limited amount of data, we shall have to accept a larger than normal risk of a false alarm. This means that our definition of what constitutes a potential signal will have to be expanded to include points that correspond to alpha levels in excess of the traditional (5%) alpha-level associated with the analysis of experimental data. Exploratory (10%) or even desperado (15% to 20%) levels will be required here. The reasons for these large alpha-levels will be illustrated in what follows.

THE *W*-RATIO TEST

Since we are concerned with the analysis of a fixed and finite data set, we dispense with the time order and arrange the k values in numerical order. While we can use either an ascending or descending order, I shall begin by using an ascending order. Let X_1 denote the smallest value, and let X_k denote the largest value, so that the following relations are satisfied.

$$X_1 \leq X_2 \leq \dots \leq X_k$$

Next, compute the k–1 successive differences between these k ordered values:

$$w_1 = X_2 - X_1 \;, \quad w_2 = X_3 - X_2 \;, \quad \dots \;, \quad w_{k-1} = X_k - X_{k-1}$$

Because the X values are arranged in numerical order the w values will have distributions that depend upon their position in the set of w values. This means that the distribution for w_1 will not be the same as that for w_2, etc. As a consequence we will need to keep track of where each difference occurs in the sequence:

$$\{ w_1, \; w_2, \; w_3, \; \dots, \; w_{k-1} \}$$

Before we can use the w values we need to make them independent of scale by dividing each of the differences by the total span of the ordered values $[\, X_k - X_1 \,]$. Thus we let:

$$W_1 = \frac{w_1}{X_k - X_1} \;, \quad W_2 = \frac{w_2}{X_k - X_1} \;, \quad etc.$$

These W_i values express each of the successive differences w_i as a proportion of the total span of the data set. As a set of proportions they will always add up to 100%. Intuitively, those proportions that are appreciably larger than $[\, 1 \,/\, (\, k - 1 \,)\,]$ will be the proportions that represent potential signals. In reality it is slightly more complex than this because of the dependency created by the ordering of the data, but this idea is the basis for the test.

We use the set of W ratios to decide which differences, if any, represent a lack of homogeneity within the set of *k* ordered values by comparing each *W* value with its corresponding set of critical values from Table 1.

Table 1: Critical Values for the *W*-Ratio Test

	alpha	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9
k = 3	1%	0.994	0.994							
	5%	0.970	0.970							
	10%	0.941	0.941							
	15%	0.913	0.913							
	20%	0.885	0.885							
	alpha	W_1	W_2	W_3						
k = 4	1%	0.934	0.919	0.934						
	5%	0.860	0.828	0.860						
	10%	0.806	0.765	0.806						
	15%	0.766	0.719	0.766						
	20%	0.733	0.681	0.733						
	alpha	W_1	W_2	W_3	W_4					
k = 5	1%	0.859	0.819	0.819	0.859					
	5%	0.766	0.707	0.707	0.766					
	10%	0.710	0.641	0.641	0.710					
	15%	0.672	0.597	0.597	0.672					
	20%	0.642	0.563	0.563	0.642					
	alpha	W_1	W_2	W_3	W_4	W_5				
k = 6	1%	0.793	0.732	0.716	0.732	0.793				
	5%	0.698	0.620	0.598	0.620	0.698				
	10%	0.646	0.560	0.536	0.560	0.646				
	15%	0.612	0.522	0.496	0.522	0.612				
	20%	0.585	0.492	0.467	0.492	0.585				
	alpha	W_1	W_2	W_3	W_4	W_5	W_6			
k = 7	1%	0.740	0.666	0.636	0.636	0.666	0.740			
	5%	0.649	0.559	0.525	0.525	0.559	0.649			
	10%	0.601	0.505	0.469	0.469	0.505	0.601			
	15%	0.570	0.470	0.434	0.434	0.470	0.570			
	20%	0.545	0.444	0.408	0.408	0.444	0.545			
	alpha	W_1	W_2	W_3	W_4	W_5	W_6	W_7		
k = 8	1%	0.698	0.612	0.574	0.566	0.574	0.612	0.698		
	5%	0.613	0.515	0.473	0.462	0.473	0.515	0.613		
	10%	0.567	0.465	0.423	0.411	0.423	0.465	0.567		
	15%	0.537	0.434	0.392	0.380	0.392	0.434	0.537		
	20%	0.515	0.410	0.368	0.357	0.368	0.410	0.515		
	alpha	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	
k = 9	1%	0.669	0.573	0.530	0.512	0.512	0.530	0.573	0.669	
	5%	0.584	0.482	0.434	0.414	0.414	0.434	0.482	0.584	
	10%	0.541	0.436	0.389	0.369	0.369	0.389	0.436	0.541	
	15%	0.514	0.406	0.360	0.341	0.341	0.360	0.406	0.514	
	20%	0.493	0.385	0.339	0.321	0.321	0.339	0.385	0.493	
	alpha	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9
k = 10	1%	0.636	0.542	0.492	0.469	0.444	0.469	0.492	0.542	0.636
	5%	0.560	0.455	0.406	0.381	0.374	0.381	0.406	0.455	0.560
	10%	0.519	0.412	0.363	0.339	0.332	0.339	0.363	0.412	0.519
	15%	0.493	0.386	0.337	0.313	0.307	0.313	0.337	0.386	0.493
	20%	0.473	0.366	0.318	0.295	0.288	0.295	0.318	0.366	0.473

Whenever an observed value for W_i exceeds one of its corresponding critical values, you may conclude that either (1) you have just observed an event having probability that is less than the alpha-level for that critical value, or else (2) the gap corresponding to W_i represents a lack of homogeneity among the k ordered values. Note that there is a symmetry in Table 1 in that W_1 and W_{k-1} will always have the same critical values. Likewise for W_2 and W_{k-2} , etc. This symmetry is what makes it possible to work with either ascending or descending orderings of the original data. An example with descending order will be given later.

The critical values in Table 1 are estimates obtained from simulation studies where the W values were repeatedly calculated in the case where there were no real differences within the original data set. The 120 estimates in Table 1 have an average standard error of 0.0006. Twelve of the estimates at the 1% level had a standard error in the range of 0.0011 to 0.0015, and one had a standard error of 0.0020. This means that most the values in Table 1 are essentially known to three digits, with a few of the 1% critical values having an error of one or two units in the third place.

EXAMPLE ONE

To illustrate the W-Ratio Test we will start with the following set of $k = 10$ values:

189, 173, 169, 190, 162, 185, 192, 166, 165, 187

These values may be examined for homogeneity by first ordering the values to obtain:

$$\{ X_1 \leq X_2 \leq \dots \leq X_{10} \} = \{ 162, 165, 166, 169, 173, 185, 187, 189, 190, 192 \}$$

Next the nine successive differences are computed for this ordered set:

$$\{ w_1, w_2, w_3, \dots, w_9 \} = \{ 3, 1, 3, 4, 12, 2, 2, 1, 2 \}$$

When divided by [192 – 162] = 30 these successive differences become the set of W values:

$$W_1 = 0.10, W_2 = 0.03, W_3 = 0.10, W_4 = 0.13, W_5 = 0.40, W_6 = 0.07, W_7 = 0.07, W_8 = 0.03, W_9 = 0.07$$

Once we have the W values we can create the graph in Figure 1. This graph provides a useful representation of the ten ordered values. The vertical scale goes from 0% to 100% and the height of each "step" is simply equal to the corresponding W value. For convenience, the units on the horizontal scale are equally spaced. This curve gives your audience a picture of what the W-Ratio Test is doing and will facilitate the interpretation of the results.

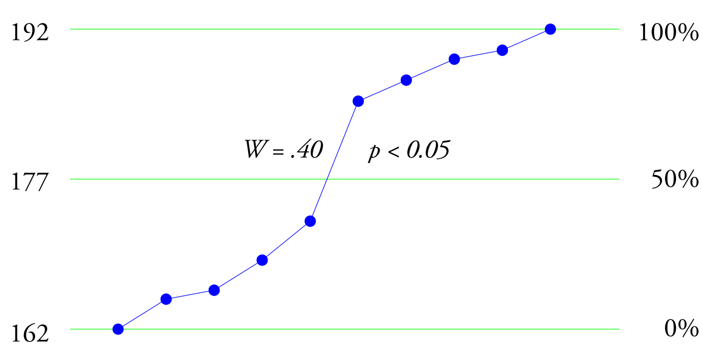

Figure 1: The Ten Ordered Values for Example One

The set of nine W values above are compared with the critical values for $k = 10$. Since W_5 exceeds its 5% critical value of 0.374 it is likely that the five largest values in Figure 1 differ from the five smallest values. This means that it would be a mistake to assume that these ten values are homogeneous. In practice the specific interpretation of this lack of homogeneity would depend upon the context for the original set of 10 values.

CHUNKY RATIOS

Because of the way they are computed the W values will have a certain chunkiness. This chunkiness will depend upon two quantities: the measurement increment, and the span of the ordered data. The measurement increment is the smallest *possible* difference between two values. In Example One the measurement increment was equal to 1 unit. The span of the ordered data was $[X_k - X_1] = 30$ units. Therefore, each value for W in Example One is equal to some multiple of $1/30 = 0.0333$ (even though they were rounded off to only two decimal places). While expressing the W values in parts per hundred will usually be sufficient to use the test, knowing the underlying chunkiness is important simply because excessively chunky ratios will create false alarms and missed signals.

When we express a W ratio using two decimal places we are implying that this value is known to parts per hundred. However, as noted above, if the span of the data contains M measurement increments, then every W value will be a multiple of $1/M$. As M gets smaller the W ratios will become more chunky. For example, when M is 20 the W values will all be multiples of 0.05, but when M is only 10 the W ratios will all be multiples of 0.1. In order to avoid the problems of excessively chunky W ratios, we prefer for M to be at least 20.

EXAMPLE TWO

For a further example of the W-Ratio Test, consider the first $k = 5$ values from the original data given in Example One above:

189, 173, 169, 190, 162

When placed in ascending order these values become:

162, 169, 173, 189, 190

This gives rise to the W values:

$W_1 = 0.250$, $W_2 = 0.143$, $W_3 = 0.571$, $W_4 = 0.036$

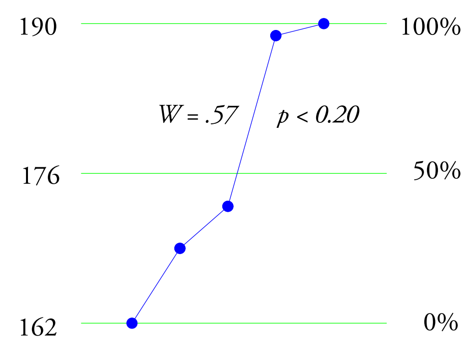

Figure 2: The Ordered Values for Example Two

These W values are compared with the critical values for $k = 5$. Since W_3 exceeds the 20% critical value of 0.563, it is within the realm of possibilities that the two largest values differ from the three smallest. This example serves to show how dramatically the critical values change depending upon the number of values in the original data set. With fewer data any analysis will become less discriminating. This is why we have to compensate by using a larger alpha-level for what we are willing to consider as a potential signal.

POWER FUNCTIONS

The power functions in Figure 3 characterize the performance of the W-Ratio Test. These curves were found by adding a known difference to one of the original data and then counting the number of times, out of 10,000, that the W-Ratio Test detected this known difference. A set of points was obtained for each value of k by changing the size of the known difference and repeating this procedure. The result being the power curves shown here.

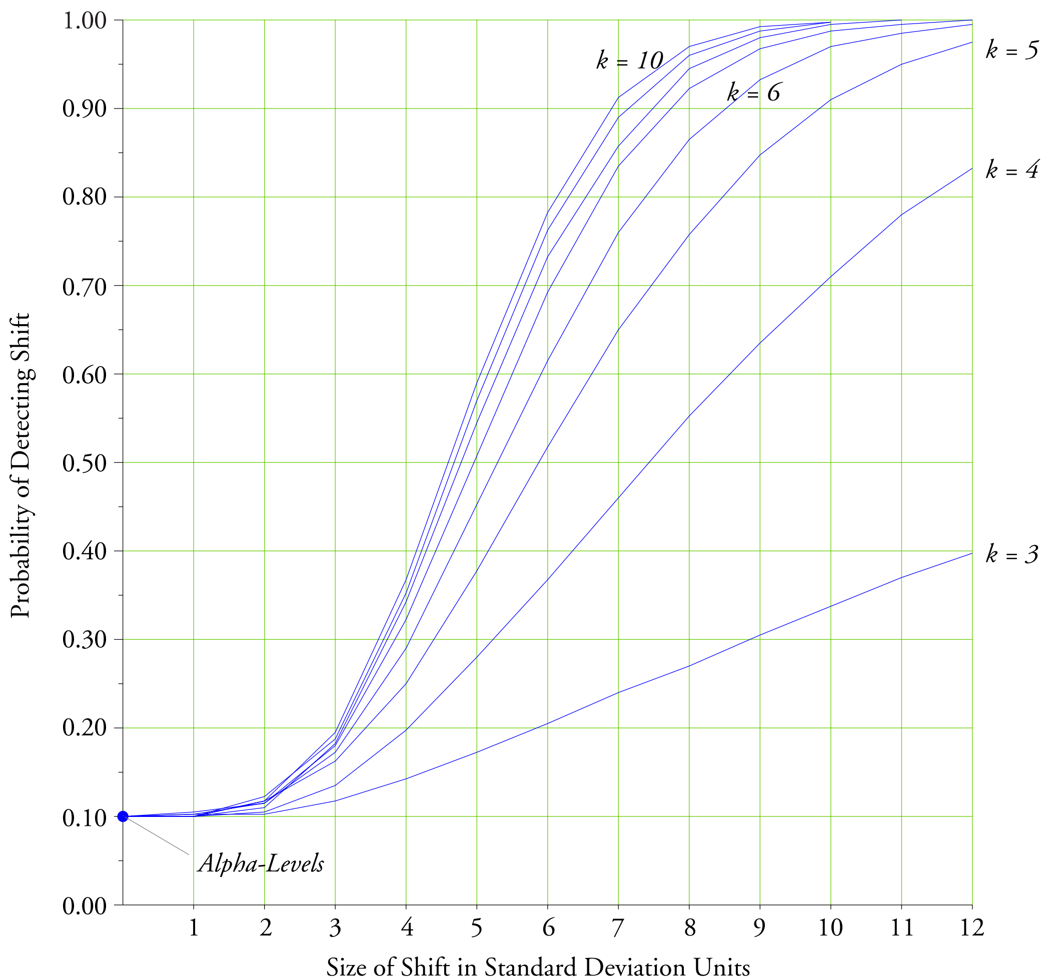

Figure 3: Power Functions for the *W*-Ratio Test

The horizontal scale in Figure 3 shows the size of the difference introduced into the data expressed in standard deviation units. The vertical scale shows the proportion of the time that that shift was detected by a given procedure. Since the left side of the graph represents the condition of no signal within the data, the beginning point of each curve represents the alpha-level for the test. Here all the curves have an alpha-level of 10%.

As we move to the right the size of the known differences increase and the probability that the various tests will detect each of these signals also increases. For example, Figure 3 shows that the *W*-Ratio Test with $k = 4$ will detect a 8 standard deviation difference about 55% of the time. The same sized shift would be detected about 75% of the time with $k = 5$, about 86% of the time with $k = 6$, and about 95% of the time with $k = 8$.

So, while Figure 3 shows the differences in power due to different amounts of data, in practice the amount of data is fixed. Thus, the practical question becomes “How does the choice of alpha-level affect the power of the *W*-Ratio Test?” To illustrate this the power function curves for $k = 10$ are shown in Figure 4 for alpha-levels of 1%, 5%, 10% and 20%. As expected, as the alpha-level increases the power function curve shifts higher.

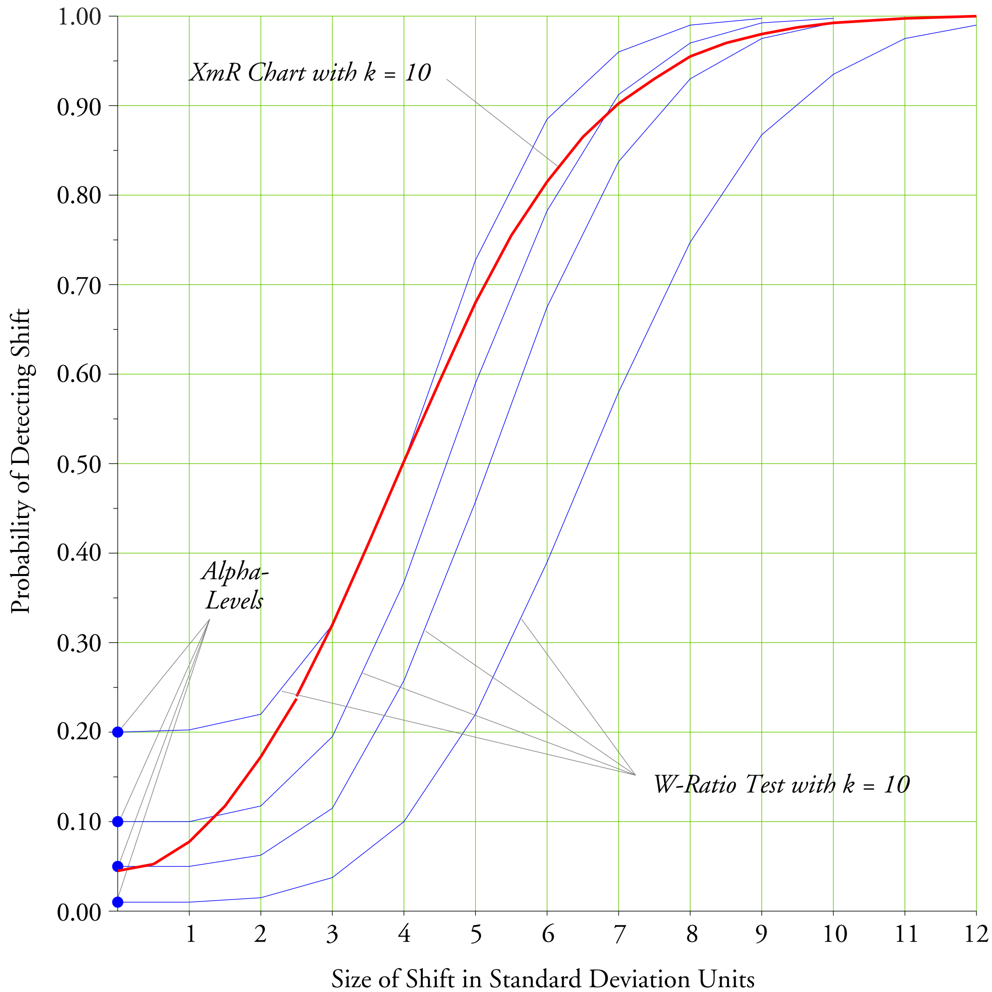

Figure 4: Why the *W*-Ratio Test Uses Alpha Levels of 10% to 20%

Since it is feasible to use $k = 10$ data with an *XmR* chart, a power function curve for an *XmR* chart is also included in Figure 4 for comparison. In over 80 years of practice we have found that the ability of the process behavior chart to detect three-sigma and larger shifts has been generally sufficient to meet our needs. To get comparable power from a *W*-Ratio Test in the three-sigma to five sigma range we will need to use an alpha-level of 20%. To get comparable power from a *W*-Ratio Test for larger shifts we will need to use a 15% or a 10% alpha-level as the cut-off for what represents a potential signal. This is the basis for recommending the use of alpha-levels of 10% to 20% when using a *W*-Ratio Test.

The reason that the *W*-Ratio Test has a greater alpha-level than the *XmR* Chart for the same power is due to the composite nature of the *W*-Ratio Test. Here we are examining $k - 1 = 9$ different test statistics looking for a potential signal. When there is no signal, each of these $k - 1$ statistics have a chance of creating a false alarm. So while the ultimate power of the *W*-Ratio test is comparable to that of an *XmR* Chart, this is achieved only by having a larger risk of a false alarm.

Figure 5 shows the power functions for the *W*-Ratio Test when $k = 3$. There we see that with an alpha-level of 20%, the W-Ratio Test with $k = 3$ will detect an 8-sigma shift about half the time.

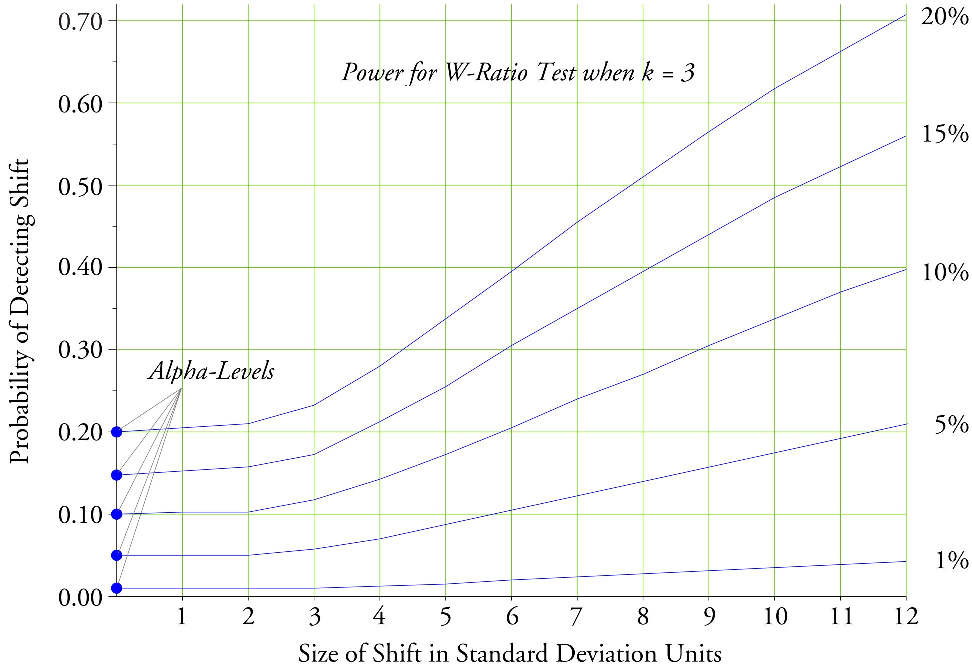

Figure 5: Power Functions for the *W*-Ratio Test when *k* = 3 for Five Alpha-Levels

EXAMPLE THREE

To illustrate the *W*-Ratio Test with $k = 3$ consider what would happen if we only had the first three of the ten values from Example One available:

189, 173, 169

Since these three values are already in descending numerical order we use them as they are. The successive differences are 16 and 4, resulting in *W*-ratios of:

$$W_1 = 0.80, \quad W_2 = 0.20$$

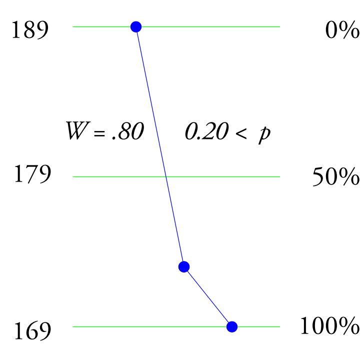

Figure 6: The Ordered Values for Example 3

In applying the *W*-Ratio Test to these data we have a Span of 20 units and a measurement increment of 1 unit, so we meet the minimum requirement for detecting a signal at an alpha-level of 15% or 20%. Since W_1 does not exceed the 20% critical value of 0.885, these three data simply do not contain enough information to allow us to detect the difference found in the earlier examples. As shown in Figure 5, the *W*-Ratio Test for $k = 3$ will reliably detect only very large signals.

SUMMARY

When you have from three to ten values and you want to know if these values can be treated as a homogeneous collection of values (so that averages, standard deviations and other statistical summaries make sense) you can use the *W*-Ratio Test. *W*-ratios that exceed their critical value are interpreted as representing a break in the ordered set of values and indicate a potential lack of homogeneity for the original set of *k* values.

Anyone desperate enough to examine fewer than 10 values for homogeneity must understand that the use of desperado alpha-levels is part of the price you have to pay to work with so few values.

Finally, the power curves for $k = 3$ show that it will always be very difficult to detect a lack of homogeneity using only three values. Differences that can be reliably detected will be so large that they will be obvious just by looking at the data.

ACKNOWLEDGMENTS

Finally, I need to thank my reviewers, James Beagle, Rip Stauffer, Dr. Henry Neave, Dr. William Woodall, and Alson Look for their many useful suggestions, comments, and advice.

CAVEATS AND COMMENTS

1. We need to pause here for a warning: Do not even think about placing ordered *X* values on an *XmR* Chart. If you should do so you will *always* find points outside the limits of the *X* chart even when the data are completely homogeneous! This means that you will have a guaranteed false alarm rate of 100%. And when false alarms are certain to occur the test becomes useless.

2. In developing any procedure involving critical values those values have to be based upon some probability model. In developing a simple test for use with individual values it is important that the probability model used will be one that will work in most situations. Over two hundred years of theory and practice has taught us that the one generic model that will provide a first-order approximation for most situations is the normal distribution. For that reason I used a normal distribution in computing the critical values given here. These values should allow you to reliably check for a lack of homogeneity in most common situations. Fortunately, we are not trying to make a statement about an experimental result where we have to prove beyond a reasonable doubt that the result is real. Rather we are simply trying to make an informed decision about the possibility of an unintentional lack of homogeneity within our data. Here the standard of proof is much lower and the exact alpha-level is not as important as the interpretation of what such a potential signal might represent.

3. Like many other statistical procedures in common use, the critical values given here are not robust. This means that with data that come from a process that produces strongly skewed data, the actual alpha-level will differ from the nominal alpha-level. As with other non-robust

procedures, the usual fix is to adjust the alpha-levels used.

Unfortunately, you cannot use any *statistics* computed from the data to determine if the data might be skewed simply because the computations would assume homogeneity. (Remember, the major cause of skewed data is a lack of homogeneity. Such skewness does not invalidate the use of the normal theory values.) Therefore, any adjustment of this procedure for skewness must be based on contextual considerations rather than computations.

How can you make this judgment call? Typically, skewed data are found when two conditions are present: There will be a boundary value for the data, and the data will pile up very close to that boundary value. If your context suggests that your limited data set might satisfy these conditions, then simply use the critical values for the next smaller alpha-level when interpreting your results.

4. "Why not use Grubb's Extreme Studentized Deviate test?" Grubb's test may be used with $k \geq 4$. (While critical values for $k = 3$ are commonly given for Grubb's test, continuity issues make its use with $k = 3$ highly questionable.) Grubb's test checks to see if the most extreme value is different from all the other values. In contrast, the W-Ratio Test checks *all* of the successive differences within the ordered data set. Because of its narrow focus, Grubb's test will have more power than the W-Ratio Test *when only one value differs from the rest*, but it will not work as well in other situations. Specifically, Grubb's test cannot detect the differences found in Examples One and Two.

Table 2: Additional Critical Values for the *W*-Ratio Test

k	*alpha*	W_1	W_2	W_3	W_4	W_5
11	1%	0.616	0.514	0.461	0.435	0.419
	5%	0.541	0.433	0.381	0.354	0.342
	10%	0.503	0.393	0.342	0.316	0.304
	15%	0.478	0.368	0.318	0.293	0.281
	20%	0.459	0.350	0.300	0.276	0.265
		W_{10}	W_9	W_8	W_7	W_6

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6
12	1%	0.596	0.492	0.439	0.408	0.392	0.389
	5%	0.525	0.417	0.363	0.333	0.317	0.315
	10%	0.488	0.379	0.326	0.298	0.283	0.280
	15%	0.464	0.355	0.303	0.276	0.261	0.258
	20%	0.447	0.337	0.287	0.260	0.247	0.243
		W_{11}	W_{10}	W_9	W_8	W_7	W_6

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6
13	1%	0.581	0.476	0.419	0.386	0.369	0.362
	5%	0.512	0.401	0.346	0.316	0.299	0.292
	10%	0.475	0.365	0.312	0.283	0.266	0.259
	15%	0.453	0.342	0.291	0.262	0.246	0.239
	20%	0.436	0.327	0.275	0.247	0.232	0.226
		W_{12}	W_{11}	W_{10}	W_9	W_8	W_7

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7
14	1%	0.568	0.461	0.404	0.370	0.351	0.340	0.337
	5%	0.500	0.388	0.333	0.301	0.283	0.273	0.270
	10%	0.465	0.354	0.300	0.270	0.252	0.244	0.240
	15%	0.443	0.332	0.280	0.251	0.233	0.225	0.222
	20%	0.426	0.317	0.265	0.237	0.220	0.212	0.210
		W_{13}	W_{12}	W_{11}	W_{10}	W_9	W_8	W_7

Table 2: Additional Critical Values for the *W*-Ratio Test (continued)

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7
15	1%	0.553	0.444	0.389	0.351	0.334	0.319	0.311
	5%	0.489	0.378	0.323	0.290	0.271	0.258	0.253
	10%	0.455	0.344	0.290	0.260	0.241	0.230	0.225
	15%	0.433	0.324	0.271	0.241	0.223	0.213	0.208
	20%	0.417	0.309	0.257	0.228	0.211	0.201	0.196
		W_{14}	W_{13}	W_{12}	W_{11}	W_{10}	W_9	W_8

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
16	1%	0.545	0.438	0.379	0.343	0.320	0.308	0.297	0.293
	5%	0.480	0.368	0.313	0.279	0.258	0.246	0.239	0.238
	10%	0.447	0.337	0.282	0.251	0.231	0.219	0.213	0.211
	15%	0.426	0.316	0.263	0.233	0.214	0.203	0.197	0.195
	20%	0.411	0.302	0.249	0.221	0.203	0.192	0.186	0.184
		W_{15}	W_{14}	W_{13}	W_{12}	W_{11}	W_{10}	W_9	W_8

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
17	1%	0.532	0.422	0.366	0.328	0.307	0.291	0.281	0.277
	5%	0.471	0.359	0.303	0.270	0.247	0.235	0.227	0.224
	10%	0.439	0.328	0.274	0.243	0.222	0.210	0.202	0.200
	15%	0.419	0.309	0.257	0.226	0.207	0.195	0.188	0.185
	20%	0.405	0.295	0.244	0.214	0.195	0.184	0.177	0.174
		W_{16}	W_{15}	W_{14}	W_{13}	W_{12}	W_{11}	W_{10}	W_9

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9
18	1%	0.527	0.417	0.361	0.323	0.298	0.281	0.273	0.266	0.266
	5%	0.463	0.352	0.296	0.261	0.241	0.226	0.217	0.213	0.212
	10%	0.432	0.322	0.268	0.236	0.215	0.203	0.194	0.190	0.188
	15%	0.413	0.304	0.251	0.219	0.200	0.188	0.180	0.176	0.175
	20%	0.399	0.290	0.239	0.208	0.189	0.178	0.170	0.166	0.165
		W_{17}	W_{16}	W_{15}	W_{14}	W_{13}	W_{12}	W_{11}	W_{10}	W_9

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9
19	1%	0.516	0.405	0.349	0.311	0.286	0.271	0.258	0.254	0.251
	5%	0.457	0.345	0.288	0.255	0.234	0.220	0.209	0.204	0.201
	10%	0.426	0.316	0.262	0.229	0.209	0.196	0.187	0.182	0.179
	15%	0.408	0.299	0.246	0.214	0.195	0.182	0.173	0.168	0.166
	20%	0.394	0.285	0.234	0.203	0.184	0.172	0.164	0.159	0.156
		W_{18}	W_{17}	W_{16}	W_{15}	W_{14}	W_{13}	W_{12}	W_{11}	W_{10}

k	*alpha*	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}
20	1%	0.509	0.396	0.336	0.301	0.276	0.260	0.248	0.240	0.236	0.236
	5%	0.452	0.339	0.282	0.248	0.227	0.211	0.202	0.194	0.192	0.189
	10%	0.421	0.311	0.257	0.224	0.204	0.190	0.180	0.174	0.171	0.169
	15%	0.403	0.294	0.241	0.210	0.190	0.177	0.168	0.162	0.159	0.157
	20%	0.389	0.281	0.229	0.199	0.180	0.167	0.159	0.153	0.150	0.149
		W_{19}	W_{18}	W_{17}	W_{16}	W_{15}	W_{14}	W_{13}	W_{12}	W_{11}	W_{10}

Table 2: Additional Critical Values for the *W*-Ratio Test (continued)